Technical Appendix of
"Non-Separable Preferences do not Rule Out Aggregate Instability under Balanced-Budget Rules : A Note"

1 Derivation of the Intertemporal Equilibrium

In order to derive the intertemporal equilibrium, let

$$\tau(t) \equiv \tilde{\tau}(K(t), l(t)) = \frac{G}{w(K(t)/l(t))l(t)}$$

and substitute $\tilde{\tau}(K, l)$ and the wage rate (11) in the first order conditions (6) and (7). Given K and λ, the system obtained can be solved to express the consumption demand and labor supply functions $c(K(t), \lambda(t))$ and $l(K(t), \lambda(t))$. Substituting the latter in the expression of the tax rate, we obtain:

$$\tilde{\tau}(K(t), l(K(t), \lambda(t))) \equiv \tau(K(t), \lambda(t)) \tag{A.1}$$

Using (10)-(11), we get the equilibrium values for the rental rate of capital $r(t)$ and the wage rate $w(t)$ with $a(t) = K(t)/l(K(t), \lambda(t))$:

$$\begin{aligned} r(t) &= Af'(a(t)) \equiv r(K(t), \lambda(t)) \\ w(t) &= A[f(a(t)) - a(t)f'(a(t))] \equiv w(K(t), \lambda(t)) \end{aligned} \tag{A.2}$$

Substituting the expressions obtained for prices, tax rate, consumption demand and labor supply in the equation of capital accumulation (5) and in the Euler equation (8), we obtain the following system of differential equations in K and λ:

$$\begin{aligned} \dot{K}(t) &= r(K(t), \lambda(t))K(t) + (1 - \tau(K(t), \lambda(t)))w(K(t), \lambda(t))l(K(t), \lambda(t)) \\ &- \delta K(t) - c(K(t), \lambda(t)) \\ \dot{\lambda}(t) &= -\lambda(t)\left[r(K(t), \lambda(t)) - \rho - \delta\right] \end{aligned} \tag{A.3}$$

2 Proof of Proposition 1

To establish the existence of a normalized steady state $(a^*, l^*, c^*, \tau^*) = (1, 1, c^*, \tau^*)$, we have to prove the existence and uniqueness of solutions A^* and B^* satisfying:

$$\begin{aligned} \delta + \rho &= A^* f'(1) \\ \tau^* &= \frac{G}{A^*[f(1) - f'(1)]} \\ c^* &= (1 - \tau^*)A^*[f(1) - f'(1)] + A^* f'(1) - \delta \\ \frac{U_{\mathcal{L}}(c, (\bar{l} - 1)/B^*)}{B^* U_c(c, (\bar{l} - 1)/B^*)} &= (1 - \tau^*)A^*[f(1) - f'(1)] \end{aligned} \tag{A.4}$$

From the first equation of (A.4), we derive that $A^* = \frac{\rho+\delta}{f'(1)}$ which gives, once substituted in the second and the third equations of (A.4), a unique τ^* and c^* such that:

$$\begin{aligned} \tau^* &= \tfrac{s(1)G}{(\rho+\delta)(1-s(1))} \\ c^* &= \tfrac{s(1)\rho+(1-\tau)(\rho+\delta)(1-s(1))}{s(1)} \end{aligned}$$

Considering A^*, τ^* and c^*, we get the following from the last equation of (A.4):

$$\tilde{g}(B) \equiv \frac{U_{\mathcal{L}}(c,(\bar{l}-1)/B)}{BU_c(c,(\bar{l}-1)/B)} = \frac{(1-\tau^*)(\rho+\delta)(1-s(1))}{s(1)} \tag{A.5}$$

Since under Assumption 1, $\lim_{B\to 0}\tilde{g}(B) = 0$ and $\lim_{B\to+\infty}\tilde{g}(B) = +\infty$, or $\lim_{B\to 0}\tilde{g}(B) = +\infty$ and $\lim_{B\to+\infty}\tilde{g}(B) = 0$, and $B\tilde{g}'(B)/\tilde{g}(B) \neq 0$, there exists a unique B^* solution of (A.5).

□

3 Proof of Lemma 1

Let us linearize (A.3) around the NSS. First, using the definitions (15) and the first order conditions (6) and (7), we get $\varepsilon_{cl} = \frac{(1-\tau)wl}{c}\varepsilon_{lc}$. Using the expression of w at the NSS given in (A.2) together with (12) and (A.4) we find $wl = K(1-s)(\delta+\rho)/s$. Since at NSS, $c = l[\rho a + (1-\tau)w]$, it follows:

$$\varepsilon_{cl} = \frac{(1-\tau)(\delta+\rho)(1-s)+s\rho}{(1-\tau)(\delta+\rho)(1-s)}\varepsilon_{lc} \tag{A.6}$$

Second, differentiating $\tau(K(t),\lambda(t))$ as given by (A.1), we obtain the elasticities of the tax rate with respect to K and λ:

$$\begin{aligned} \varepsilon_{\tau k} = \tfrac{d\tau}{dK}\tfrac{K}{\tau} &= -\tfrac{(1-\tau)s}{\sigma}\tfrac{[\sigma\Delta\varepsilon_{cc}+\sigma-s]}{(1-\tau)\sigma\Delta\varepsilon_{cc}+\tau(s-\sigma)} \\ \varepsilon_{\tau\lambda} = \tfrac{d\tau}{d\lambda}\tfrac{\lambda}{\tau} &= -\tfrac{(1-\tau)(\sigma-s)\varepsilon_{cc}}{(1-\tau)\sigma\Delta\varepsilon_{cc}+\tau(s-\sigma)}\left(\tfrac{1}{\varepsilon_{cc}} - \tfrac{1}{\varepsilon_{lc}}\right) \end{aligned}$$

Third, using (15), the Implicit Function Theorem gives the partial derivatives of the functions $c(K(t),\lambda(t))$ and $l(K(t),\lambda(t))$ evaluated at the NSS:

$$\begin{aligned} \tfrac{dc}{dK} &= \tfrac{c}{K\Delta\varepsilon_{cl}}\left(\tfrac{s}{\sigma} - \tfrac{\tau\varepsilon_{\tau k}}{1-\tau}\right), \quad \tfrac{dc}{d\lambda} = -\tfrac{c}{\lambda\Delta}\left[\tfrac{1}{\varepsilon_{ll}} - (1-\tfrac{\tau\varepsilon_{\tau\lambda}}{1-\tau})\tfrac{1}{\varepsilon_{cl}} + \tfrac{s}{\sigma}\right] \\ \tfrac{dl}{dK} &= \tfrac{l}{K\Delta\varepsilon_{cc}}\left(\tfrac{s}{\sigma} - \tfrac{\tau\varepsilon_{\tau k}}{1-\tau}\right), \quad \tfrac{dl}{d\lambda} = \tfrac{l}{\lambda\Delta}\left[(1-\tfrac{\tau\varepsilon_{\tau\lambda}}{1-\tau})\tfrac{1}{\varepsilon_{cc}} - \tfrac{1}{\varepsilon_{lc}}\right] \end{aligned}$$

with $\Delta = \frac{1}{\varepsilon_{cc}}\left(\frac{1}{\varepsilon_{ll}} + \frac{s}{\sigma}\right) - \frac{1}{\varepsilon_{cl}\varepsilon_{lc}}$. From these results and (A.2) we also derive at the NSS:

$$\begin{aligned} \tfrac{dr}{dK} &= -\tfrac{r(1-s)}{K\sigma}\left[1 - \tfrac{1}{\Delta\varepsilon_{cc}}\left(\tfrac{s}{\sigma} - \tfrac{\tau\varepsilon_{\tau k}}{1-\tau}\right)\right], \quad \tfrac{dr}{d\lambda} = \tfrac{r(1-s)}{\lambda\Delta\sigma}\left[(1-\tfrac{\tau\varepsilon_{\tau\lambda}}{1-\tau})\tfrac{1}{\varepsilon_{cc}} - \tfrac{1}{\varepsilon_{lc}}\right] \\ \tfrac{dw}{dK} &= \tfrac{ws}{K\sigma}\left[1 - \tfrac{1}{\Delta\varepsilon_{cc}}\left(\tfrac{s}{\sigma} - \tfrac{\tau\varepsilon_{\tau k}}{1-\tau}\right)\right], \quad \tfrac{dw}{d\lambda} = -\tfrac{ws}{\lambda\Delta\sigma}\left[(1-\tfrac{\tau\varepsilon_{\tau\lambda}}{1-\tau})\tfrac{1}{\varepsilon_{cc}} - \tfrac{1}{\varepsilon_{lc}}\right] \end{aligned}$$

Finally, linearizing the system (A.3) around the NSS, using (A.6) and the above results, gives:

$$\begin{aligned}
\frac{d\dot{K}}{dK} &= \rho - \frac{(\delta+\rho)(1-s)}{s}\left\{\tau\left[\varepsilon_{\tau k} + \frac{s}{\sigma}\left[1 - \frac{1}{\Delta\varepsilon_{cc}}\left(\frac{s}{\sigma} - \frac{\tau\varepsilon_{\tau k}}{1-\tau}\right)\right]\right] - \frac{1-\tau}{\Delta\varepsilon_{cc}}\left(\frac{s}{\sigma} - \frac{\tau\varepsilon_{\tau}k}{1-\tau}\right)\right\} \\
&- \frac{(1-\tau)(1-s)(\delta+\rho)}{s\Delta\varepsilon_{cl}}\left(\frac{s}{\sigma} - \frac{\tau\varepsilon_{\tau k}}{1-\tau}\right) \\
\frac{d\dot{K}}{d\lambda} &= \frac{(1-\tau)(1-s)(\delta+\rho)K}{s\Delta\lambda}\left[\frac{1}{\varepsilon_{ll}} + \frac{s}{\sigma} - \left(1 - \frac{\tau\varepsilon_{\tau\lambda}}{1-\tau}\right)\frac{1}{\varepsilon_{cl}}\right] + (1-\tau)\left[\left(1 - \frac{\tau\varepsilon_{\tau\lambda}}{1-\tau}\right)\frac{1}{\varepsilon_{cc}} - \frac{1}{\varepsilon_{lc}}\right] \\
&+ \frac{(\delta+\rho)(1-s)K}{s\lambda}\left\{\tau\left[\Delta\varepsilon_{\tau\lambda} - \frac{s}{\sigma}\left[\left(1 - \frac{\tau\varepsilon_{\tau\lambda}}{1-\tau}\right)\frac{1}{\varepsilon_{cc}} - \frac{1}{\varepsilon_{lc}}\right]\right]\right\} \\
\frac{d\dot{\lambda}}{dK} &= -\frac{\lambda(\delta+\rho)(1-s)}{K\sigma}\left[\Delta + \frac{1}{\Delta\varepsilon_{cc}}\left(\frac{s}{\sigma} - \frac{\tau\varepsilon_{\tau k}}{1-\tau}\right)\right] \\
\frac{d\dot{\lambda}}{d\lambda} &= -\frac{(\delta+\rho)(1-s)}{\Delta\sigma}\left[\left(1 - \frac{\tau\varepsilon_{\tau\lambda}}{1-\tau}\right)\frac{1}{\varepsilon_{cc}} - \frac{1}{\varepsilon_{lc}}\right]
\end{aligned}$$

After tedious computations and straightforward simplifications, using (A.6), the expressions of $\varepsilon_{\tau k}$, $\varepsilon_{\tau\lambda}$ as given above, we get the following characteristic polynomial:

$$\mathcal{P}(\lambda) = \lambda^2 - \mathcal{T}\lambda + \mathcal{D} = 0 \tag{A.7}$$

with

$$\mathcal{T} = \frac{d\dot{K}}{dK} + \frac{d\dot{\lambda}}{d\lambda} = \rho - \frac{(\rho+\delta)(1-s)\tau}{\sigma\tau - s - (1-\tau)\sigma\varepsilon_{cc}\left[\frac{1}{\varepsilon_{cc}}\frac{1}{\varepsilon_{ll}} - \frac{1}{\varepsilon_{cl}}\frac{1}{\varepsilon_{lc}}\right]}$$

and

$$\begin{aligned}
\mathcal{D} &= \frac{d\dot{K}}{dK}\frac{d\dot{\lambda}}{d\lambda} - \frac{d\dot{K}}{d\lambda}\frac{d\dot{\lambda}}{dK} \\
&= \frac{(\rho+\delta)(1-s)\varepsilon_{cc}\left[[(1-\tau)(\rho+\delta)(1-s)+s\rho]\left[(1-\tau)\left(\frac{1}{\varepsilon_{cc}} - \frac{1}{\varepsilon_{lc}} + \frac{1}{\varepsilon_{ll}} - \frac{1}{\varepsilon_{cl}}\right) - \tau\right] + \tau(1-\tau)(\rho+\delta)(1-s)\left(\frac{1}{\varepsilon_{cc}} - \frac{1}{\varepsilon_{lc}}\right)\right]}{s\sigma\left[\sigma\tau - s - (1-\tau)\sigma\varepsilon_{cc}\left[\frac{1}{\varepsilon_{cc}}\frac{1}{\varepsilon_{ll}} - \frac{1}{\varepsilon_{cl}}\frac{1}{\varepsilon_{lc}}\right]\right]}
\end{aligned}$$

Local indeterminacy requires $\mathcal{T} < 0$ and $\mathcal{D} > 0$. A necessary condition for $\mathcal{T} < 0$ is $\tau > \underline{\tau}$ with:

$$\underline{\tau} = \frac{\frac{s}{\sigma} + \varepsilon_{cc}\left(\frac{1}{\varepsilon_{cc}}\frac{1}{\varepsilon_{ll}} - \frac{1}{\varepsilon_{cl}}\frac{1}{\varepsilon_{lc}}\right)}{1 + \varepsilon_{cc}\left(\frac{1}{\varepsilon_{cc}}\frac{1}{\varepsilon_{ll}} - \frac{1}{\varepsilon_{cl}}\frac{1}{\varepsilon_{lc}}\right)}$$

In the linearly homogeneous case, the elasticities are given by:

$$\begin{gathered}
\varepsilon_{lc} = -\varepsilon_{cc}\frac{(1-\alpha)}{\alpha}, \quad \varepsilon_{cl} = -\varepsilon_{cc}\frac{(1-\alpha)}{\alpha}\frac{(1-\tau)(\delta+\rho)(1-s)+s\rho}{(1-\tau)(1-s)(\rho+\delta)}, \\
\varepsilon_{ll} = \varepsilon_{cc}\frac{(1-\alpha)^2[(1-\tau)(\rho+\delta)(1-s)+s\rho)}{\alpha^2(1-\tau)(\rho+\delta)(1-s)}
\end{gathered} \tag{A.8}$$

and $\frac{1}{\varepsilon_{cc}\varepsilon_{ll}} - \frac{1}{\varepsilon_{lc}\varepsilon_{cl}} = 0$, while we derive with the JR formulation:

$$\begin{aligned}
\frac{1}{\varepsilon_{cc}} &= \theta\frac{c - \gamma\frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma}{c - \frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma} - \gamma(1-\gamma)\frac{\frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma}{c - \gamma\frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma}, \quad & \frac{1}{\varepsilon_{ll}} &= \theta\frac{\frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma}{c - \frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma} + \chi, \\
\frac{1}{\varepsilon_{cl}} &= \frac{\frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma}{c - \gamma\frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma}\left[\theta\frac{c - \gamma\frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma}{c - \frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma} - \gamma\right], & \frac{1}{\varepsilon_{lc}} &= \theta\frac{c - \gamma\frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma}{c - \frac{(l/B)^{1+\chi}}{1+\chi}c^\gamma} - \gamma,
\end{aligned} \tag{A.9}$$

Using these expressions and the relationship between ε_{cl} and ε_{lc} at NSS given by equation (A.6), we derive:

$$\frac{\frac{(l/B)^{1+\chi}}{1+\chi}c^{\gamma-1}}{1 - \gamma\frac{(l/B)^{1+\chi}}{1+\chi}c^{\gamma-1}} = \frac{(1-\tau)(\delta+\rho)(1-s)}{(1-\tau)(\delta+\rho)(1-s)+s\rho} \equiv \mathcal{C}(\tau)$$

Re-arranging this equation gives:

$$\frac{(l/B)^{1+\chi}}{1+\chi}c^{\gamma-1} = \frac{\mathcal{C}(\tau)(1+\chi)}{1+\chi+\gamma\mathcal{C}(\tau)}$$

Then, the following expressions hold:

$$\frac{c-\gamma\frac{(l/B)^{1+\chi}}{1+\chi}c^{\gamma}}{c-\frac{(l/B)^{1+\chi}}{1+\chi}c^{\gamma}} = \frac{1+\chi}{1+\chi-(1-\gamma)\mathcal{C}(\tau)}, \quad \frac{\frac{(l/B)^{1+\chi}}{1+\chi}c^{\gamma}}{c-\gamma\frac{(l/B)^{1+\chi}}{1+\chi}c^{\gamma}} = \frac{(1+\chi)\mathcal{C}(\tau)}{1+\chi-(1-\gamma)\mathcal{C}(\tau)}$$

The elasticities rewrite therefore:

$$\begin{aligned} \frac{1}{\varepsilon_{cc}} &= \theta\frac{1+\chi}{1+\chi-(1-\gamma)\mathcal{C}(\tau)} - \gamma(1-\gamma)\frac{\mathcal{C}(\tau)}{1+\chi}, & \frac{1}{\varepsilon_{ll}} &= \theta\frac{(1+\chi)\mathcal{C}(\tau)}{1+\chi-(1-\gamma)\mathcal{C}(\tau)} + \chi, \\ \frac{1}{\varepsilon_{lc}} &= \theta\frac{1+\chi}{1+\chi-(1-\gamma)\mathcal{C}(\tau)} - \gamma, & \frac{1}{\varepsilon_{cl}} &= \frac{\mathcal{C}(\tau)(\tau)}{\varepsilon_{lc}}, \end{aligned} \tag{A.10}$$

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